CONTENTS

NOTE: All diagrams are deliberately approximate. This is to ensure that the answers cannot be easily calculated by simply measuring the lengths and angles from the diagrams, and also to test the students' ability to accurately draw diagrams from given information.

MATHEMATICS PACK 3

By Sharanjit Uppal
In collaboration with Harry Jivenmukta

ALGEBRA 1

Algebra is defined as being the study of numbers using general symbols. The symbols used are usually letters of the alphabet. Questions involving algebra are not as difficult as they might first appear, as there are a few steps which should be applied to all algebraic equations which, if followed, make the questions very easy.

SOLVING SIMPLE ALGEBRAIC EQUATIONS

Below is a very simple algebraic equation in which the letter 'p' stands for a number. The question accompanying this sort of equation is usually find the value of 'p'.

⌘ $p + 18 = 23$

Even if you have never done any algebra before it is probably obvious that 'p' must have a value of five as $5 + 18 = 23$.

However, if the equation had been more complex the value of 'p' would not have been so obvious. Therefore to answer the above question you need to know the first fact that is applied to algebraic equations which is that you want the 'p' on the left of the equal sign and you want all the other terms on the right side of the equal sign.

When moving any term from one side of the equal sign to the other you must always do the opposite of what the term was initially doing. For the above equation you need to move the 18 to the other side of the equal sign. At the moment the 18 is being added onto the 'p' so when you move it you have to do the opposite: subtract 18.

$p = 23 - 18$ (Note the sign change in front of the eighteen)

$p = 5$

SIMPLIFYING EQUATIONS INVOLVING LIKE TERMS AND THEN SOLVING

Sometimes the algebraic equation you are asked to solve has like terms. Find the value of 'p' in the equation below giving your answer to three significant figures.

⌘ $q^3 + 4p^2 - p^2 = q^3 + 36$

In the equation above the $4p^2$ and the $-p^2$ are like terms as they both involve p^2. The two q^3 terms are also like terms. Before trying to solve these types of equations you should first combine the like terms in order to simplify the equation as follows.

1) Combine the p^2 terms. $4p^2 - p^2 = 3p^2$

2) Substitute this into the equation.

$q^3 + 3p^2 = q^3 + 36$

3) Combine the q^3 terms. This is slightly more difficult to do as the two q^3 terms are not on the same side of the equal sign. Therefore you first bring one of the q^3 terms to the other side remembering to do the opposite of what the term was initially doing. If we move the q^3 term to the right of the equal sign the q^3 term was initially being added to the p^2 term so we now need to subtract it.

q3 - q3 = 0 (The q^3 term has therefore been eliminated from the equation)

4) By substituting the result of the above step into the equation the equation now reads:

$3p^2 = 36$

5) Now that the equation has been simplified you solve it by getting the equation to read p = ...

To do this you first need to deal with any numbers in front of the p^2 term; in this case the number three. You want to get rid of the three from the left side of the equal sign and you do that by moving it to the other side of the equal sign. Remember when moving any term to the other side you need to do the opposite of what the term was initially doing. Initially the three was being multiplied to the p^2 so to do the opposite you need to divide by three.

i) Divide both sides by three:

$3p^2 \mid 3 = {}^{36}/_{3}$

$p^2 = 12$

There is one final step that you need to do before the equation reads p = ...

At the moment the 'p' is being squared and we need to get rid of the square. To get rid of the square we move the square to the other side of the equal sign and to do that we need to do the opposite. The opposite of squaring something is to square root it.

ii) Square root both sides.

$\sqrt{p^2} = \sqrt{12}$

$p = \sqrt{12}$

p = 3.46 (to three significant figures)

SIMPLIFYING EQUATIONS INVOLVING BRACKETS AND THEN SOLVING

Algebraic equations involving brackets are solved in exactly the same way as described above except you first must multiply everything inside the brackets by everything outside of the brackets as for the first example below. However if there are two sets of brackets you multiply everything in the first bracket by everything in the second bracket as in the second example below.

EXAMPLE 1

Find the value of 'r' in the following equation.

⌘ ${}^{1}/_{3}$ (6r + 3r - 6) = 18

1) First combine any like terms: 6r + 3r = 9r

${}^{1}/_{3}$ (9r - 6) = 18

2) Multiply everything inside the brackets by everything outside the brackets.

${}^{1}/_{3} \cdot 9r = 3r$

${}^{1}/_{3} \cdot -6 = -2$

3r -2 = 18

3) Move the two to the other side of the equal sign. The two is being subtracted from the 3r so you need to add two to both sides to move it. By adding two to both sides the equation reads:

3r = 20

To find 'r' you need to move the three by dividing both sides of the equation by three.

$r = {}^{20}/_{3}$

$r = 6\ {}^{2}/_{3}$

EXAMPLE 2

Find the value of 's' in the following equation.

⌘ $(s - 5)(s - 8) = 0$

1) Multiply everything in the first bracket by everything in the second bracket.

$s \cdot s = s^2$

$s \cdot -8 = -8s$

$-5 \cdot s = -5s$

$-5 \cdot -8 = 40$

The equation reads: $s^2 -8s -5s + 40 = 0$

2) Combine the like terms. $-8s - 5s = -13s$

3) Substitute the results of step two into the equation.

$s^2 - 13s + 40 = 0$

The equation formed is a quadratic equation and solving quadratic equations is covered later.

ALGEBRAIC FRACTIONS

Simplifying algebraic fractions is exactly the same as simplifying ordinary fractions.

⌘ Multiplication of algebraic fractions

When multiplying numerical fractions you simply multiply everything on the top rows of the fractions together and you put your answer over what you get when you multiply everything on the bottom rows of the fractions together. You do exactly the same for algebraic fractions.

For example simplify $2t^3 / u \cdot tu^2 / 6$

$2t^3 \cdot tu2 = 2t^4u^2$

$u \cdot 6 = 6u$

$2t^3 / u \cdot tu^2 / 6 = 2t^4u^2 / 6u$

Now you can simplify the fraction as 'u' divides into u^2 to give 'u' and $^2/_6$ can be simplified to give $^1/_3$.

$2t^4u^2 / 6u = t^4u / 3$

⌘ Division of algebraic fractions.

Division of algebraic fractions is done in exactly the same way as division of numerical fractions. First you invert the second fraction. You then multiply the two fractions together as shown above.

For example simplify $b^3 / 4 \mid b^4 / 5$

1) $b^3 / 4 \cdot 5 / b^4$

2) $5 \cdot b^3 = 5b^3$

$4 \cdot b^4 = 4b^4$

$b^3 / 4 \cdot 5 / b^4 = 5b^3 / 4b^4$

3) Simplify: $5b^3 / 4b^4 = {}^5/_{4b}$

⌘ Addition of algebraic fractions

Adding algebraic fractions is similar to adding numerical fractions.

First you find a common denominator: the bottom of each fraction should have the same number and/or letters.

You then add whatever is on the top row of each fraction together.

For example calculate $^{a}/_{2} + {}^{3a}/_{b}$

1) Find a common denominator.

To find the common denominator you multiply the bottom rows of each fraction together.

Common denominator = 2 · b = 2b

Then multiply the top of the first fraction by the bottom of the second fraction. You then multiply the top of the second fraction by the bottom of the first fraction.

a · b = ab   3a · 2 = 6a

$^{a}/_{2} + {}^{3a}/_{b} = {}^{ab}/_{2b} + {}^{6a}/_{2b}$

$= (ab + 6a) / 2b$

⌘ Subtraction of algebraic fractions

This is done using the same method as above except once you have found a common denominator you don't add together what is on the top rows of the fractions you subtract instead.

For example calculate $6/a - a^3/2$

Common denominator = a · 2 = 2a

Then multiply the top of the first fraction by the bottom of the second fraction. You then multiply the top of the second fraction by the bottom of the first fraction.

$6 \cdot 2 = 12$ $a^3 \cdot a = a^4$

$6/a - a^3/2 = 12/2a - a^4/2a$

$= (12 - a^4) / 2a$

SIMPLIFYING EQUATIONS BY FACTORISATION

Below is an equation which can be simplified by factorisation.

$4x^3 + 6xy - 2x^2$

When you factorise an equation you take out the highest common factor for all the terms in the equation. If you look at the above equation you should notice that the number 2 is the highest common factor for all the terms so you place the number 2 outside of the brackets as shown below.

$4x^3 + 6xy - 2x^2 = 2(2x^3 + 3xy - x^2)$

Remember to alter the values inside the brackets accordingly. There is another common factor in the equation which can be placed outside of the brackets; the common factor is 'x' as there is at least one 'x' in each of the terms.

$2(2x^3 + 3xy - x^2) = 2x(2x^2 + 3y - x)$

The equation above cannot be simplified further and is your final answer. You may think it is possible to take out another 'x' from the above equation, by factorisation, as the terms $2x^2$ and $-x$ both have at least one 'x' in them, however the term 3y does not have an 'x' so it is not possible to factorise further.

⌘WORKED EXAMPLE 1

Simplify the equation $4a - 12ab + 24a + 8a^2 = 0$

1) First combine any like terms:

$4a + 24a = 28a$

Substitute the result of this step into the original equation.

$28a - 12ab + 8a^2 = 0$

2) Take out the highest common factor/factors for all the terms.

The number four is the highest common factor for all the terms.

$28a - 12ab + 8a^2 = 4(7a - 3ab + 2a^2)$

The letter 'a' is common to all the terms.

$4(7a - 3ab + 2a^2) = 4a(7 - 3b + 2a)$

⌘WORKED EXAMPLE 2

Find the value of T in the equation below, given S has a value of seven.

$T^3 - S^3 - 4T^2 + S^2 = 0$

Combine like terms.

$-3T^3 - S^3 + S^2 = 0$

Substitute seven into the above equation in place of S.

$-3T^3 - 7^3 + 7^2 = 0$

$-3T^3 - 343 + 49 = 0$

$-3T^3 - 294 = 0$

Now move the T^2 term to the other side of the equal sign. Remember when you do this you need to do the opposite of what the term was initially doing. The T^2 term is being subtracted so when you move it you add it to both sides.

$-294 = 0 + 3T^3 = 3T^3$

You now need to get rid of the three in front of the T^3. At the moment the three is being multiplied so you need to divide both sides by three. Then cube root both sides to find T.

$^{-249}/_{3} = T^3$

$\sqrt[3]{-83} = \sqrt[3]{T^3}$

T = -4.36 (to three significant figures)

QUESTIONS ON ALGEBRA 6

EXERCISE 1

1) Find the value of 'b' in the following equations.

a) $b - 15 = 32$

b) $b + 23 = 11.5$

c) $45 - b + 6b = -3$

d) $18b - 36 = -2b$

e) $b^2 = 81$ (given that $b > 0$)

f) $b^2 + 16b^2 = 68$ (given that $b > 0$)

g) $27 - b^3 = b^3$

2) Find the value of 'k' in the following equations.

a) $2(k - 1) = 0$

b) $3k(2k + 9k) = 7k^2 + 1$

c) $2k^2(k + 5) = 10(k^2 + 1)$

d) $k^3 - 4k^3 + 7 = k(k^2 - 2k^2)$

3) Simplify the following:

a) $^{b}/_{a} \times {}^{b^2}/_{b}$

b) $^{mn}/_{p} \times {}^{p}/_{n}$

c) $^{d}/_{ef}$ | $^{d}/_{e}2$

d) $^{st}/_{r}$ | $^{t}/_{r}2$

e) $^{g}/_{h} + {}^{f}/_{g}$

f) $^{pq}/_{r} + {}^{p}/_{q}$

g) $^{ce}/_{d} - {}^{c}/_{2}$

4) Simplify the following by factorisation.

a) $2ab^2 + 4abc$

b) $3ad + 5af$

c) $3b^2 + ab + b$

d) $g^2h - g^3f$

e) $2ad + 2bd - 4ac$

f) $3xy + 6y^2 + 12x^2$

g) $e^3f^2 - e^2gf + e^2f^2$

h) $14x^2y + 7y + 7x^2$

INEQUALITIES 7

There are worked examples below showing you how to solve inequalities but it is important you first know the meanings of the following symbols.

SYMBOL	MEANING
>	Greater than.
<	Less than.
≥	Greater than or equal to.
≤	Less than or equal to.

Inequalities can be solved in the same way as algebraic equations; there is however one rule you need to know.

Rule: When you wish to divide/multiply both sides of an inequality by a negative number the original equality sign MUST be reversed. For example if the original equality sign is > it would become < and if the original equality sign is ≤ it would become ≥.

WORKED EXAMPLE 1

Solve the inequality $2y - 5 > 0$

The best way to solve an inequality is to treat it like a normal algebraic equation but do not forget the above rule. If the inequality was a normal equation it would read as follows:

$2y - 5 = 0$

From our knowledge of algebra we know that the first thing we need to do is get rid of the 5 so that the equation reads 2y = ...

As the five is being subtracted from the 2y you should know that you need to do the opposite to remove it: you need to add five to both sides of the equation. That gives you

$2y = 5$

We then need to get rid of the two in front of the 'y' so that we can find the values 'y' takes. At the moment the two is being multiplied by 'y' so to remove it we do the opposite: we divide both sides by two. It is important that you now refer back to the rule stated earlier. The rule says that the equality sign has to be reversed if you divide by a negative number but as the two is not negative it does not apply in this case. We can therefore continue with the same equality sign.

$y = {}^{5}/_{2}$

At this point you can now replace the equal (=) sign with the equality sign.

$y > 2.5$

This means y can have any value as long as it is greater than 2.5.

Once you become more comfortable with inequality problems it is not necessary to use the = sign and you should use the equality sign. The examples below show that this does not make a difference to the method.

WORKED EXAMPLE 2

Solve the inequality $-2z \leq 7z + 14$

This inequality has more than one 'z' term so you should begin by combining the like terms. You can do this by moving the 7z to the left of the equality sign. As the 7z is being added you move it by subtracting 7z from both sides:

$-2z - 7z \leq 14$

$-9z \leq 14$

To find the values that 'z' can take we need to get rid of the -9 and we do this by dividing both sides by -9. It is important that you remember that dividing by a negative number reverses the equality sign.

$\leq$ becomes $\geq$

$z \geq {}^{14}/_{-9}$

$z \geqq -1.56$ (to three significant figures).

This means 'z' can be any value greater than or equal to -1.56.

WORKED EXAMPLE 3

Solve the inequality $3x^3 - 26 \geq 5x^3 + 9x^3 + 10$

First combine the like terms.

$3x^3 - 5x^3 - 9x^3 - 26 \geq 10$

$-11x^3 - 26 \geq 10$

$-11x^3 \geq 10 + 26$

$-11x^3 \geq 36$

$x^3 \leq {}^{36}/_{-11}$

$x^3 \leq -3.2727$

$x \leq \sqrt[3]{-3.2727}$

$x \leqq -1.48$ (to three significant figures)

WORKED EXAMPLE 4

Solve the inequality $x^2 > 9$

To find 'x' you need to square root both sides and you should remember that when you square root a number you get a negative answer as well as a positive answer.

$x = \sqrt{9}$

$x = 3$ and $x = -3$

It is easy to work out which equality sign to use if you think of a number line.

-5 -4 -3 -2 -1 0 1 2 3 4 5

It is now possible to see that if the numbers highlighted are squared they would be greater than 9.

Therefore $x > 3$ and $x < -3$

QUESTIONS ON INEQUALITIES

EXERCISE 2

1) Solve the following inequalities. If the answer can not be given exactly it should be given to three significant figures.

a) $6x + 4 < 5$

b) $15x - 7 \geq 9$

c) $10x + 5 < 0$

d) $36x + 18 < 16$

e) $-73x + 75 > 0$

f) $14x - 39 \leq 37$

g) $-21x + 13 > -6$

h) $22 - 45x > 63$

2) Solve the following inequalities. If the answer can not be given exactly it should be given correct to three significant figures.

a) $50 + 48y \leq -45y$

b) $3y - 9 > 12y$

c) $10y + 8 - 4y > -17y$

d) $26 - 23y < 46y$

e) $13y + 16 < 22y + 5y$

f) $-19y - 8 \geq -29$

3) Solve the following inequalities. If the answer can not be given exactly it should be given to three significant figures.

a) $z^2 \geq 16$

b) $z^2 > 81$

c) $z^2 < 25$

d) $z^2 < 49$

e) $-46 + z^2 \leq -42$

f) ${}^1/_4 z^2 \leq 16$

g) $-5z^2 - 11 \geq -20$

4) Solve the following inequalities. If the answer can not be given exactly it should be given to three significant figures.

a) $4x^2 - 2x^2 < 2$

b) $3x^3 + 5 - 7x^3 < 6x^3 + 2x^3$

c) $15x^2 + x^3 \geq 7 + x^3$

d) $3x(-2x + 9x) > 42$

QUADRATIC EQUATIONS 10

A quadratic equation is simply an algebraic equation which has an x^2 term in it. Quadratic equations are slightly harder to solve than the algebraic equations we looked at earlier. However, there are three different methods which make solving quadratic equations easy and these are:

⌘ FACTORISATION

⌘ THE QUADRATIC FORMULA

⌘ COMPLETING THE SQUARE

Each of the above methods are described in detail below with a worked example.

⌘ Factorisation

This method involves adjusting the quadratic equation given so that it can be placed in two sets of brackets. All quadratic equations can be expressed as follows:

$ax^2 + bx + c$

QUESTION: Solve the quadratic equation $3x^2 - 7x - 20 = 0$

In this question a = 3, b = -7 and c = -20.

Before attempting to solve any quadratic equation you should first write down the following:

$(\quad)(\quad) = 0$

We then begin by looking at the $3x^2$ term. You need to consider all the different ways in which you can get the answer $3x^2$ by multiplying together two other 'x' terms.

$?x \cdot ?x = 3x^2$

The only way in which you can get three as your answer from multiplying two numbers together is to multiply three by one.

$3x \cdot 1x = 3x^2$ This is the same as writing $3x \cdot x = 3x^2$

You now fill in part of the brackets as follows:

$(3x \quad)(x \quad) = 0$

To complete the brackets you now need to find two numbers which when multiplied together give -20 and also give -7x once the brackets are multiplied out. -20 can be made in the following ways:

$-1 \cdot 20 = -20$	$-2 \cdot 10 = -20$	$-4 \cdot 5 = -20$
$1 \cdot -20 = -20$	$2 \cdot -10 = -20$	$4 \cdot -5 = -20$

Fill in the brackets with each of the above combinations and then multiply out the brackets until you find the combination that gives you -7x.

$(3x - 1)(x + 20) = 3x^2 + 60x - x - 20 = 3x^2 + 59x - 20$

$(3x + 20)(x - 1) = 3x^2 - 3x + 20x - 20 = 3x^2 + 17x - 20$

$-1 \cdot 20 = -20$

$(3x + 1)(x - 20) = 3x^2 - 60x + x - 20 = 3x^2 - 59x - 20$

$(3x - 20)(x + 1) = 3x^2 + 3x - 20x - 20 = 3x^2 - 17x - 20$

$1 \cdot -20 = -20$

$(3x - 2)(x + 10) = 3x^2 + 30x - 2x - 20 = 3x^2 + 28x - 20$

$(3x + 10)(x - 2) = 3x^2 - 6x + 10x - 20 = 3x^2 + 4x - 20$

$-2 \cdot 10 = -20$

$(3x + 2)(x - 10) = 3x^2 - 30x + 2x - 20 = 3x^2 - 28x - 20$

$(3x - 10)(x + 2) = 3x^2 + 6x - 10x - 20 = 3x^2 - 4x - 20$

$(3x - 4)(x + 5) = 3x^2 + 15x - 4x - 20 = 3x^2 - 11x - 20$

$(3x + 5)(x - 4) = 3x^2 - 12x + 5x - 20 = 3x^2 - 7x - 20$

$2 \cdot -10 = -20$

$-4 \cdot 5 = -20$

The above combination has given you -7x so $3x^2 - 7x - 20 = 0 = (3x + 5)(x - 4)$

We can now solve the quadratic equation as follows.

To find the values of 'x' you simply use the rules you applied in algebra.

$(3x + 5)(x - 4) = 0$

Therefore $3x + 5 = 0$ and $x - 4 = 0$

$3x = -5$ $x = 4$

$x = -5/3$

As can be seen from above the factorisation method can be fairly lengthy where as the quadratic formula is much quicker to use. The quadratic formula also allows you to find the value of 'x' when it is not an integer.

⌘ The Quadratic Formula

The quadratic formula can be applied to any quadratic equation by simply substituting into the formula correctly.

QUADRATIC FORMULA $x = [-b \pm \sqrt{(b^2 - 4ac)}] \mid 2a$

QUESTION: Solve the quadratic equation $5x^2 + 17x - 5 = 0$ using the quadratic formula.

You should first write down the value of a, b and c using $ax^2 + bx + c$.

$a = 5$, $b = 17$ and $c = -5$

Then substitute these values into the quadratic formula.

$x = [-17 \pm \sqrt{\{17^2 - (4 \times 5 \times -5)\}}] \mid (2 \times 5)$

$x = [-17 \pm \sqrt{389}] \mid 10$

$x = 0.272$ or $x = -3.67$ (to three significant figures).

⌘ Completing The Square

Consider the equation $x^2 - 6x + 3 = 0$

To solve this by completing the square you first write down the following:

$(x \quad)^2$

You complete the bracket by dividing 'b' by two. $b = -6$ so $b/2 = -3$

$(x - 3)^2 \quad = 0$

You then compare the above to the quadratic equation given. Notice that when you square -3 it gives you 9 but you need to get 3. You therefore have to subtract 6.

Therefore $(x - 3)^2 - 6 = 0$

$(x - 3)^2 = 6$

$x = \sqrt{6} + 3$ $x = -\sqrt{6} + 3$

$x = 5.45$ and $x = 0.551$

QUESTIONS ON QUADRATIC EQUATIONS

EXERCISE 3

1) Solve the following quadratic equations using the factorisation method.

a) $x^2 + 4x + 3 = 0$

b) $x^2 + 5x + 6 = 0$

c) $x^2 - x -2 = 0$

d) $x^2 - x - 6 = 0$

e) $x^2 - 8x + 15 = 0$

f) $x^2 + 5x + 4 = 0$

g)$2x^2 - 2x -12 = 0$

h) $5x^2 + 17x + 6 = 0$

i) $6x^2 - 17x + 12 = 0$

2) Solve the following quadratic equations using the quadratic formula. If the answer can not be given exactly it should be given to three significant figures.

a) $7x^2 + 12x + 5 = 0$

b) $2x^2 - 15x + 4 = 0$

c) $3x^2 - 8x + 1 = 0$

d) $x^2 + 11x - 5 = 0$

e) $-5x^2 + 4x + 2 = 0$

f) $-x^2 - 6x + 2 = 0$

3) Solve the following quadratic equations using the completing the square method. If the answer can not be given exactly then it should be given to three significant figures.

a) $x^2 + 10x + 11 = 0$

b) $x^2 - 8x - 5 = 0$

c) $x^2 + 16x - 13 = 0$

d) $x^2 - 5x + 6 = 0$

SIMULTANEOUS EQUATIONS

The following are a pair of simultaneous equations.

$3x + y = 6$ and $x - 8y = 15$

It is possible to solve these simultaneous equations to find the value of 'x' and 'y'.

Below is the method you should use to find 'x' and 'y'.

First number each equation.

(1) $3x + y = 6$ (2) $x - 8y = 15$

You now need to make the coefficients in front of either the 'x' or the 'y' equal in both equations. (The coefficient is the number in front of the letter).

We will choose to make the coefficients in front of the 'x' equal. To do this you should notice that the coefficient of 'x' in equation one is 3 and the coefficient of 'x' in equation two is 1. It is easy to make the coefficient of 'x' in equation two also equal 3. You can do this by multiplying the whole of equation two by 3.

(2) multiplied by 3: $3x - 24y = 45$ (3)

We can now eliminate the 'x' terms by subtracting one equation from the other.

Equation (1) - Equation (3):

$$
\begin{aligned}
(1)\ 3x + y &= 6 \\
(3)\ 3x - 24y &= 45 \quad - \\
25y &= -39
\end{aligned}
$$

We are now able to find the value of 'y' by dividing both sides of the equation by 25.

$25y = -39$

$y = -{}^{39}/_{25}$

$y = -1.56$

As we know the value of 'y' it is possible to find the value of 'x'. You can find 'x' by substituting the value of 'y' into any of the above equations. We have chosen to substitute into equation (1).

(1) $3x + y = 6$

(1) $3x - 1.56 = 6$

$3x = 6 + 1.56$

$3x = 7.56$

$x = {}^{7.56}/_{3}$

$x = 2.52$

You can check that the value found for 'x' and 'y' are correct by substituting into any of the original equations. We will substitute into equation (2).

(2) $x - 8y = 15$

(2) $2.52 - (8 \times -1.56) = 15$

QUESTIONS ON SIMULTANEOUS EQUATIONS

EXERCISE 4

1) Solve the following pairs of simultaneous equations. If your answer can not be given exactly it should be given to three significant figures.

a) $2m + n = 13$

$7m - n = 19$

b) $12p - 4q = 16$

$11p - 8q = 12$

c) $9r - 3s = 0$

$2r + 5s = 4$

d) $21d + 25e = 0$

$3d + 5e = -10$

2) Max has a bag of marbles. The marbles are of two different weights: x and y.

When Max has seven marbles of weight x and three marbles of weight y the total weight is 5Kg.

When Max has two marbles of weight x and two marbles of weight y the total weight is 3 Kg.

a) Write down a pair of simultaneous equations relating x and y from the information given in the question.

b) Use your pair of simultaneous equations to find the weight of marble x and marble y. Give your answers to three significant figures.

3) Solve the following simultaneous equations giving your answers to three significant figures.

a) $15u + 3v = 11$

$u - 6v = 14$

b) $5f - e = -25$

$5e - f = 10$

c) $-a + 3b = 20$

$6a + 5b = -80$

VARIATION 15

Before learning how to solve variation questions it is important that you learn the terminology and symbols used.

SYMBOL	MEANING
α	...is proportional to...
k or any other letter	constant

It is also important that you learn how to form equations from the information given in a question and that you can interpret equations.

Example 1: Interpret the following proportionality: $D \alpha kS$ where 'k' is a constant.

The above means D is proportional to S.

Example 2: Interpret the following: $G \alpha {}^{k}/_{F}$ where 'k' is a constant.

The above means G is inversely proportional to F.

Example 3: Interpret the following: $R \alpha kT^3$ where 'k' is a constant.

The above means R is proportional to T cubed.

Example 4: Interpret the following proportionality: $P \alpha {}^{m}/_{Q^2}$ where 'm' is a constant.

The above means that P is inversely proportional to the square of Q.

Example 5: Given that H represents height and A represents the age of a person, interpret the following. $H \alpha kA$ where 'k' is a constant.

We already know that the symbol highlighted stands for "is proportional to" so the above means H is proportional to A. i.e. The height of a person is proportional to their age.

Example 6: Given that L represents the number of hours of sunlight and Y represents crop yield interpret the following. $L \alpha kY$ where 'k' is a constant.

The above means that L is proportional to Y i.e. the number of hours of sunlight is proportional to the crop yield.

Example 7: Using V to represent the volume of water consumed and C to represent the cost form a proportionality given that C is directly proportional to the cube of V. Use 'k' to represent the constant.

Answer: $C \alpha kV^3$

Example 8: Using D to represent the distance travelled in miles and V to represent the volume of petrol left in a car form a proportionality given that D is inversely proportional to the square root of V.

Answer: $D \alpha {}^{k}/_{\sqrt{V}}$

WORKED EXAMPLE 1

The temperature of a room is inversely proportional to the square root of the time the air conditioner has been switched on for. Given that the temperature (H) of a room is 18 ^{o}C when the air conditioner has been on for thirty minutes find the temperature of the room when the air conditioner has been on for twenty minutes. Then calculate how long the air conditioner has been switched on if the temperature of the room is 15 ^{o}C.

In this question you should use H to represent the temperature and T to represent the time.

Answer:

⌘ The first step is always to write down the proportionality from the information given in the question.

"The temperature of a room is inversely proportional to the square root of the time the air conditioner has been switched on for."

$H \alpha \, ^{k}/_{\sqrt{T}}$ where 'k' is a constant.

⌘ The second step is to replace α with an equal sign (=).

$H = \, ^{k}/_{\sqrt{T}}$

⌘ You then calculate the value of 'k' by substituting the information given in the question into the equation.

"The temperature (H) of a room is 18 ^{o}C when the air conditioner has been on for thirty minutes."

H = 18 and T = 30

$H = \, ^{k}/_{\sqrt{T}}$

$18 = \, ^{k}/_{\sqrt{30}}$

$k = 18 \text{ x } \sqrt{30}$

k = 98.59 (to two decimal places)

⌘ Now that you have found the value of 'k' you substitute it into the original equation.

$H = \, ^{98.59}/_{\sqrt{T}}$

⌘ It is now possible to answer the question, as you substitute the information given into the above equation and perform the calculation.

"Find the temperature of the room when the air conditioner has been on for twenty minutes."

i.e. find H when T = 20

$H = \, ^{98.59}/_{\sqrt{20}}$

Temperature of the room (H) = 22.0 ^{o}C (to one decimal place)

"Then calculate how long the air conditioner has been switched on if the temperature of the room is 15^{o}C."

i.e. find T when H = 15

$15 = \, ^{98.59}/_{\sqrt{T}}$

$15 \text{ x } \sqrt{T} = 98.59$

$\sqrt{T} = \, ^{98.59}/_{15} = 6.573$

$T = 6.573^{2}$

Time the air conditioner has been on (T) = 43 minutes (to the nearest minute)

QUESTIONS ON VARIATION

EXERCISE 5

1) Interpret the following proportionalities. The first one has been done as an example for you.

a) $y \; \alpha \; k / \sqrt[3]{x}$ (k is a constant)

The above means 'y' is inversely proportional to the cube root of 'x'.

b) $r \; \alpha \; ks$

c) $u \; \alpha \; k\sqrt{v}$

d) $c \; \alpha \; kd^4$

e) $a \; \alpha \; {}^{k}/_{b}$

f) $i \; \alpha \; {}^{k}/_{j^3}$

2) Write down the proportionality for the following statements. The first one has been done as an example for you.

a) H is directly proportional to the square of G.

$H \; \alpha \; kG^2$ (where k is a constant).

b) Q is directly proportional to the fifth root of P.

c) V is inversely proportional to the square root of W.

d) M is directly proportional to NP.

e) Y is inversely proportional to the cube of Z.

f) L is directly proportional to the square of M.

3) 'x' is directly proportional to the square of 'y'. When 'x' has a value of 7.5 'y' has a value of -3. Find the value of:

a) 'x' when y has a value of 5.5

b) 'y' when 'x' has a of 2.

Give your answers to three significant figures.

4) 'C' is inversely proportional to the fourth root of 'V'. When C has a value of ${}^{1}/_{16}$ V has a value of 81. Find the value of:

a) 'C' when 'V' has a value of 16.

b) 'V' when 'C' has a value of 4.

5) D is directly proportional to the cube of E. When E has a value of 2.5 D has a value of 31.25.

a) Find the value of D when E has a value of 6.

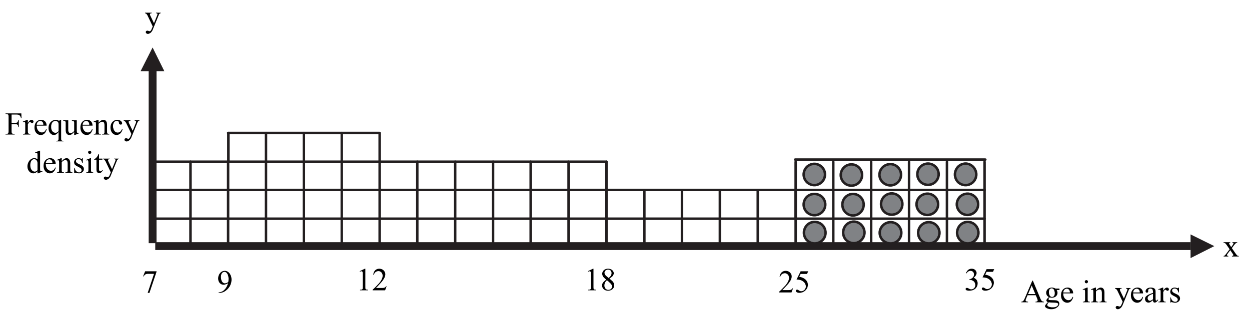
b) Find the value of E when D has a value of 128.

HISTOGRAMS 18

When looking at a histogram it may appear to be the same as a bar chart; the difference between a histogram and a bar chart is that the width of each of the bars does not have to be equal in a histogram where as it does in a bar chart. When you look at a bar chart you interpret the results by looking at the height of each bar where as with a histogram it is the area of each bar that is important.

<u>WORKED EXAMPLE</u>

The histogram below shows the number of people who have at least one filling, who visit a particular dentist, ranging from the ages of seven to thirty five. In the range of twenty five to thirty five there are forty five people who have at least one filling. Find the number of people in each of the other age groups with at least one filling.



To answer this question you must first find the number of people represented by one square unit. To do this you use the information given in the question.

"In the range of twenty five to thirty five there are forty five people who have at least one filling."

The forty five people are represented by fifteen square units (these have been highlighted for you). To find the number of people represented by one square unit you divide forty five by fifteen.

Number of people represented by one square unit = $^{45}/_{15}$

Number of people represented by one square unit = 3

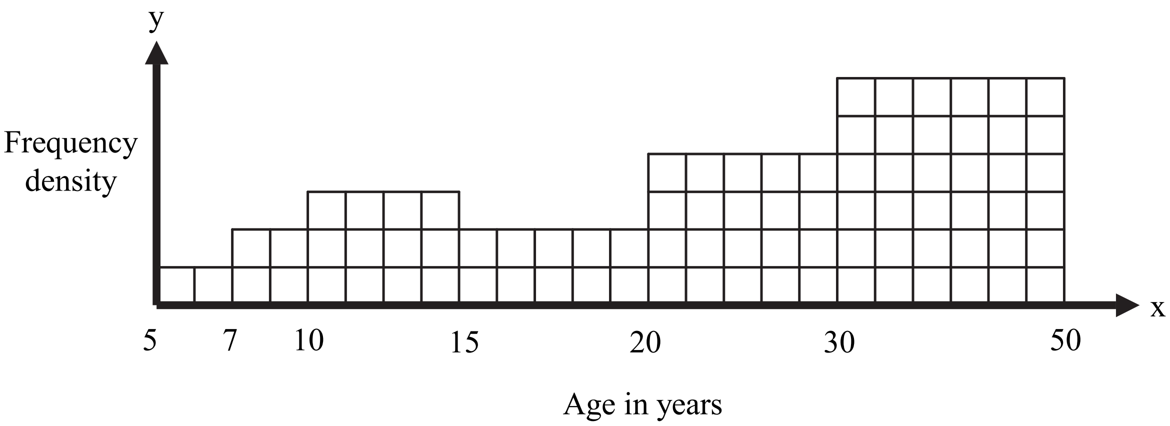
You can now find the number of people in the other age groups, with at least one filling, by counting the number of square units (or finding the area) for each range and multiplying that number by three.

Age	Area	No. of people with at least one filling
7 - 9	2 x 3 = 6	6 x 3 = 18
9 - 12	4 x 4 = 16	16 x 3 = 48
12 - 18	6 x 3 = 18	18 x 3 = 54
18 - 25	5 x 2 = 10	10 x 3 = 30
25 - 35	5 x 3 = 15	15 x 3 = 45

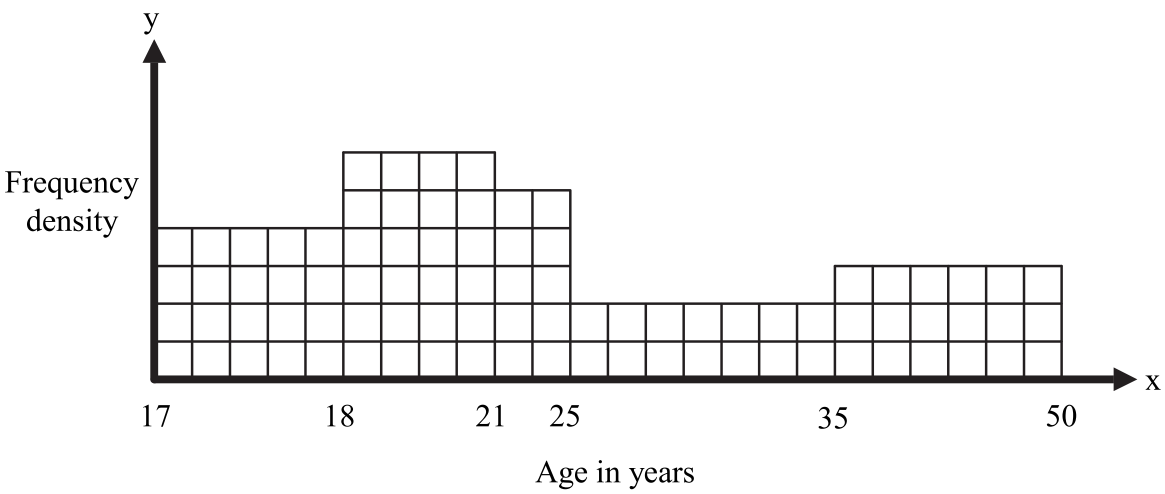
QUESTIONS ON HISTOGRAMS

EXERCISE 6

1) The histogram below shows the number of people who wear spectacles, ranging from the ages of five to fifty. In the range of fifteen to twenty there are five people who wear spectacles. Find the number of people in each of the other age groups who wear spectacles.



2) The histogram below shows the number of people at a car auction who passed their driving test on their first attempt, ranging from the ages of seventeen to fifty. In the range of seventeen to eighteen there are forty people who passed first time. Find the number of people in each of the other age groups who passed their driving test on their first attempt.



CUMULATIVE FREQUENCY CURVES

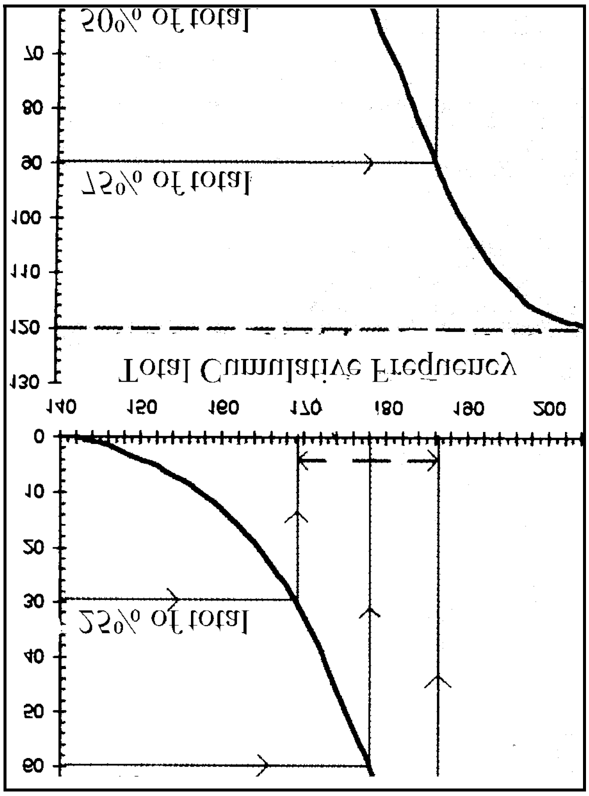
Cumulative frequency curves show the spread of data and allow you to calculate the median (middle value in a set of data) which is also known as the 50th percentile. Cumulative frequency curves also allow you to calculate the interquartile range which is the range of data between the upper quartile (75th percentile) and lower quartile (25th percentile).

You may be asked to find the median in which case you do the following:-

1) You look along the y-axis and divide the number at the top by two.

2) Whatever your answer is, you draw a horizontal line parallel to the x-axis until you reach the curve.

3) You then draw a vertical line which is parallel to the y-axis until you reach the x-axis. The value on the x-axis is the value of the median.



If asked to find the interquartile range you follow the steps given below:

1) You find the value of the upper quartile which is also known as the 75th percentile. You do this by multiplying the greatest value the curve reaches on the y-axis by 0.75. You then draw a horizontal line across from this value until you reach the curve and you then draw a vertical line until you reach the x-axis. The value on the x-axis is the value of the upper quartile.

2) You then find the value of the lower quartile range in the same way as above except you multiply by 0.25 instead of 0.75.

3) Once you have your upper and lower quartile values to find the interquartile range you take the lower quartile from the upper quartile:

Interquartile range = Upper quartile - Lower quartile

QUESTION

EXERCISE 7

1a) Calculate the value of the median for the cumulative frequency curve drawn above.

b) Calculate the value of the interquartile range for the cumulative frequency curve shown above using the method given.

ANSWER PAGE

EXERCISE 1

1a) $b = 47$

b) $b = -11.5$

c) $b = -9.6$

d) $b = 1.8$

e) $b = 9$

f) $b = 2$

g) $b = 2.38$

2a) $k = 1$

b) $k = 0.196$

c) $k = 1.71$

d) $k = 1.52$

3a) b^2/a

b) m

c) d^2/e^3f

d) rs

e) $(g^2 + hf)/gh$

f) $(pq^2 + rp)/rq$

g) $(2ce - dc)/2d$

4a) $2ab(b + 2c)$

b) $a(3d + 5f)$

c) $b(3b + a + 1)$

d) $g^2(h - gf)$

e) $2(ad + bd - 2ac)$

f) $3(xy + 2y^2 + 4x^2)$

g) $e^2f(ef - g + f)$

h) $7y(2x^2 + 1 + x^2)$

EXERCISE 2

1a) $x < {}^1/_6$

b) $x \geq 1.07$

c) $x < -0.5$

d) $x < -0.0556$

e) $x < 1.03$

f) $x \leq 5.43$

g) $x < 0.905$

h) $x < 0.911$

2a) $y \leq -0.538$

b) $y < -1$

c) $y > -0.727$

d) $y > 0.377$

e) $y > 1.14$

f) $y \leq 1.11$

3a) $z \leq -4, z \geq 4$

b) $z > 9, z < -9$

c) $z < 5, z > -5$

d) $z < 7, z > -7$

e) $z \leq 2, z \geq -2$

f) $z \leq 8, z \geq -8$

g) $z \leq 1.34, z \geq -1.34$

4a) $x < 1, x > -1$

b) $x > 0.747$

c) $x \geq 0.683, x \leq -0.683$

d) $x > 1.41, x < -1.41$

EXERCISE 3

1a) $x = -1, x = 3$

b) $x = -2, x = -3$

c) $x = 2, x = 1$

d) $x = 3, x = -2$

e) $x = 5, x = 3$

f) $x = -4, x = -1$

g) $x = 3, x = -2$

h) $x = -3, x = -{}^2/_5$

i) $x = {}^4/_3, x = 1.5$

2a) $x = -0.714, x = -1$

b) $x = 7.22, x = 0.277$

c) $x = 2.54, x = 0.131$

d) $x = -65.3, x = -11.4$

e) $x = -0.348, x = 1.15$

f) $x = -6.32, x = 0.317$

3a) $x = -1.26, x = -8.74$

b) $x = 8.58, x = -0.583$

c) $x = 0.775, x = -16.8$

d) $x = 3, x = 2$

EXERCISE 4

1a) $m = 3.56, n = 5.89$

b) $p = 1.54, q = 0.615$

c) $r = 0.235, s = 0.706$

d) $d = 8\,{}^1/_3, e = -7$

2a) $7x + 3y = 5$

$2x + 2y = 3$

b) $x = 0.125$Kg,

$y = 1.38$Kg

3a) $u = 1.16, v = -2.14$

b) $f = -4.79, e = 1.04$

c) $b = 1.74, a = -14.8$

EXERCISE 5

1b) 'r' is directly propotional to 's'.

c) 'u' is directly proportional to the square root of 'v'.

d) 'c' is directly proportional to 'd' to the power four.

e) 'a' is inversely proportional to 'b'.

f) 'i' is inversely proportional to the cube of 'j'.

2b) $Q \propto k\,{}^5\sqrt{P}$

c) $V \propto {}^k/_{\sqrt{w}}$

d) $M \propto kNP$

e) $Y \propto {}^k/_{Z^3}$

f) $L \propto kM^2$

3a) $x = 25.2$

b) $y = 1.55$ or $y = -1.55$

4a) $C = {}^3/_{32}$

b) $V = 4.83 \times 10^{-6}$

5a) $D = 432$

b) $E = 4$

EXERCISE 6

1)

Ages	No.
5-7	1
7-10	2
10-15	6
20-30	10
30-50	18

2)

Ages	No.
18-21	48
21-25	20
25-35	28
35-50	36

EXERCISE 7

1a) Median = 178

b) Interquartile range = 60